PHYC 511

Problem Session 10 — 04/13/2018

(1) Problem 7.15, parts (a) and (b), Jackson.

(2) Problem 7.20, Jackson.

(1) (a) The transit time of a pulse over distance dz is:

$$dt(\omega) = \frac{dz}{v_g(\omega)} \quad , \quad v_g(\omega) = \frac{d\omega}{dk}$$

Thus:

$$dt(\omega) = \frac{dk}{d\omega} dz = dz \frac{1}{c} \frac{d}{d\omega}\left[\omega \underbrace{\sqrt{1-\frac{\omega_p^2}{\omega^2}}}_{= 1-\frac{\omega_p^2}{2\omega^2} + O\left(\frac{\omega_p}{\omega}\right)^4}\right]$$

Then:

$$dt(\omega) \simeq \frac{dz}{c}\left(1+\frac{\omega_p^2}{2\omega^2}\right) = \frac{dz}{2c\omega^2}\frac{n_e(z)e^2}{\varepsilon_0 m_e} + \frac{dz}{c} \Rightarrow t(\omega) = \int_{z=0}^{R} dt(\omega) \Rightarrow$$

$$\boxed{t(\omega) \simeq \frac{R}{c} + \frac{1}{2\varepsilon_0 c\omega^2 m_e}\int_0^R n_e(z)\, dz}$$

(b) We have:

$$\delta\theta = \frac{1}{2}\left|\phi_+ - \phi_-\right|$$

Where $\phi_\pm$ denote phase pick up by $\pm$ helicity states. For propagation through distance dz:

$$\phi_\pm = \frac{\omega}{c}\sqrt{\frac{\varepsilon_\pm}{\varepsilon_0}}\, dz \simeq dz\frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2} \mp \frac{\omega_B}{\omega}\frac{\omega_p^2}{\omega^2}\hat{b}\cdot\hat{n}} = dz\frac{\omega}{c}\left[1 - \frac{\omega_p^2}{\omega^2} \mp \frac{1}{2}\frac{\omega_p^2}{\omega^2}\frac{e\vec{B}\cdot\hat{n}}{m_e\omega B_{\parallel}} + O\left(\frac{\omega_p}{\omega}\right)^4\right]$$

Therefore:

$$\delta\theta(z \to z+dz) \simeq \frac{1}{2} dz \frac{\omega}{c} \frac{\omega_p^2}{\omega^2} \frac{e B_{\parallel}(z)}{\omega m_e} = \frac{e^3}{2c\omega^3 \varepsilon_0 m_e^2} n_e(z) B_{\parallel}(z)\, dz$$

The full linear polarization rotation is obtained to be:

$$\boxed{\delta\theta(0 \to R) = \frac{e^3}{2\varepsilon_0 c\omega^3 m_e^2} \int_0^R n_e(z) B_{\parallel}(z)\, dz}$$

(2) (a) Each frequency ω is associated with its propagation constant $k(\omega) = \frac{\omega}{c} n(\omega)$. In one dimension, a plane wave of frequency ω can propagate along $+x$ or $-x$ direction. Hence, a plane was has the following general form:

$$A(\omega)\, e^{i[k(\omega) x - \omega t]} + B(\omega)\, e^{i[k(\omega) x + \omega t]}$$

Superposing monochromatic plane waves then generates a general wave as follows:

$$\boxed{U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \left[A(\omega)\, e^{i\frac{\omega}{c} n(\omega) x} + B(\omega)\, e^{-i\frac{\omega}{c} n(\omega) x} \right] e^{-i\omega t}}$$

(b) If $U(x,t)$ is real, we will have:

$$U^*(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \left[A^*(\omega)\, e^{-i\frac{\omega}{c} n^*(\omega) x} + B^*(\omega)\, e^{i\frac{\omega}{c} n^*(\omega) x} \right] e^{i\omega t} \underset{\text{using } \omega = -\omega'}{=}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega' \left[A^*(-\omega')\, e^{i\frac{\omega'}{c} n^*(-\omega') x} + B^*(-\omega')\, e^{-i\frac{\omega'}{c} n^*(-\omega') x} \right] e^{-i\omega' t} =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \left[A^*(-\omega)\, e^{i\frac{\omega}{c} n^*(-\omega) x} + B^*(-\omega)\, e^{i\frac{\omega}{c} n^*(-\omega) x} \right] e^{-i\omega t}$$

Comparing with $U(x,t)$ in part (a), we get:

$$\boxed{A(\omega) = A^*(-\omega), \quad B(\omega) = B^*(-\omega), \quad n(\omega) = n^*(-\omega) \Rightarrow n(-\omega) = n^*(\omega)}$$

(c) From part (a), we have:

$$U(0,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \left[A(\omega) + B(\omega) \right] e^{-i\omega t}$$

$$\frac{\partial U}{\partial x}(0,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} i\frac{\omega}{c} n(\omega) \left[A(\omega) - B(\omega) \right] e^{-i\omega t}$$

By taking the inverse Fourier transform of these two relations and solving for $A(\omega)$ and $B(\omega)$, we find:

$$\boxed{\begin{aligned} A(\omega) &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \; e^{i\omega t} \left[U(0,t) - \frac{ic}{\omega n(\omega)} \frac{\partial U}{\partial x}(0,t) \right] \\ B(\omega) &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \; e^{i\omega t} \left[U(0,t) + \frac{ic}{\omega n(\omega)} \frac{\partial U}{\partial x}(0,t) \right] \end{aligned}}$$